Summary of key points

1 A **probability distribution** fully describes the probability of any outcome in the sample space.

2 The sum of the probabilities of all outcomes of an event add up to 1. For a random variable X, you can write $\sum P(X = x) = 1$ for all x.

3 You can model X with a **binomial distribution, B(n, p)**, if:
- there are a fixed number of trials, n
- there are two possible outcomes (success or failure)
- there is a fixed probabillity of success, p
- the trials are independent of each other.

4 If a random variable X has the binomial distribution B(n, p) then its probability mass function is given by

$$P(X = r) = \binom{n}{r} p^r (1 - p)^{n-r}$$

Summary of key points

1 The null hypothesis, H_0, is the hypothesis that you assume to be correct.

2 The alternative hypothesis, H_1, tells us about the parameter if your assumption is shown to be wrong.

3 Hypothesis tests with alternative hypotheses in the form H_1: $p <$ … and H_1: $p >$ … are called one-tailed tests.

4 Hypothesis tests with an alternative hypothesis in the form H_1: $p \neq$ … are called two-tailed tests.

5 A critical region is a region of the probability distribution which, if the test statistic falls within it, would cause you to reject the null hypothesis.

6 The critical value is the first value to fall inside of the critical region.

7 The actual significance level of a hypothesis test is the probability of incorrectly rejecting the null hypothesis.

8 For a two-tailed test the critical region is split at either end of the distribution.

9 For a two-tailed test, halve the significance level at each end you are testing.

Example 6

The random variable $X \sim B(20, 0.4)$. Find:

a $P(X \leqslant 7)$ **b** $P(X < 6)$ **c** $P(X \geqslant 15)$

a $P(X \leqslant 7) = 0.4159$

Use $n = 20$, $p = 0.5$ and $x = 7$. You can use tables or your calculator.

b $P(X < 6) = P(X \leqslant 5)$
$= 0.1256$

Online Use the binomial **cumulative distribution** function on your calculator. You want to find $P(X \leqslant 7)$, not $P(X = 7)$. On some calculators, this is labelled 'Binomial CD'.

X can only take whole number values, so $P(X < 6) = P(X \leqslant 5)$.

c $P(X \geqslant 15) = 1 - P(X \leqslant 14)$
$= 1 - 0.9984$
$= 0.0016$

A single observation is taken from a binomial distribution B(6, p). The observation is used to test H_0: $p = 0.35$ against H_1: $p > 0.35$.

a Using a 5% level of significance, find the critical region for this test.

b State the actual significance level of this test.

a Assume H_0 is true then $X \sim B(6, 0.35)$
$P(X \geqslant 4) = 1 - P(X \leqslant 3) = 1 - 0.8826$
$= 0.1174$
$P(X \geqslant 5) = 1 - P(X \leqslant 4) = 1 - 0.9777$
$= 0.0223$
The critical region is 5 or 6.

Use tables or your calculator to find the first value of x for which $P(X \geqslant x) < 0.05$.

$P(X \geqslant 4) > 0.05$ but $P(X \geqslant 5) < 0.05$ so 5 is the critical value.

b The actual significance level is the probability of incorrectly rejecting the null hypothesis:
P(reject null hypothesis) $= P(X \geqslant 5)$
$= 0.0223$
$= 2.23\%$

This is the same as the probability that X falls within the critical region.

A random variable X has binomial distribution B(40, p). A single observation is used to test H_0: $p = 0.25$ against H_1: $p \neq 0.25$.

a Using the 2% level of significance, find the critical region of this test. The probability in each tail should be as close as possible to 0.01.

b Write down the actual significance level of the test.

a Assume H_0 is true then $X \sim B(40, 0.25)$
Consider the lower tail:
$P(X \leqslant 4) = 0.0160$
$P(X \leqslant 3) = 0.0047$
Consider the upper tail:
$P(X \geqslant 19) = 1 - P(X \leqslant 18) = 1 - 0.9983$
$= 0.0017$
$P(X \geqslant 18) = 1 - P(X \leqslant 17) = 1 - 0.9884$
$= 0.0116$
The critical regions are $0 \leqslant X \leqslant 3$ and $18 \leqslant X \leqslant 40$.

$P(X \leqslant 3)$ is closest to 0.01 so 3 is the critical value for this tail.

Watch out Read the question carefully. Even though $P(X \geqslant 18)$ is greater than 0.01 it is still the closest value to 0.01. The critical value for this tail is 18.

b The actual significance level is $0.0047 + 0.0116 = 0.0163 = 1.63\%$

Online Use technology to explore the locations of the critical values for each tail in this example.

Question 19

Sue throws a fair coin 15 times and records the number of times it shows a head.

(a) State the distribution to model the number of times the coin shows a head.
(2)

Find the probability that Sue records

(b) exactly 8 heads,
(2)

(c) at least 4 heads.
(2)

Sue has a different coin which she believes is biased in favour of heads. She throws the coin 15 times and obtains 13 heads.

(d) Test Sue's belief at the 1% level of significance. State your hypotheses clearly.
(6)
(Total 12 marks)

Question 6

It is known from past records that 1 in 5 bowls produced in a pottery have minor defects. To monitor production a random sample of 25 bowls was taken and the number of such bowls with defects was recorded.

(a) Using a 5% level of significance, find critical regions for a two-tailed test of the hypothesis that 1 in 5 bowls have defects. The probability of rejecting, in either tail, should be as close to 2.5% as possible.
(6)

(b) State the actual significance level of the above test.
(1)

At a later date, a random sample of 20 bowls was taken and 2 of them were found to have defects.

(c) Test, at the 10% level of significance, whether or not there is evidence that the proportion of bowls with defects has decreased. State your hypotheses clearly.
(7)
(Total 14 marks)

Question 14

A company claims that a quarter of the bolts sent to them are faulty. To test this claim the number of faulty bolts in a random sample of 50 is recorded.

(a) Give two reasons why a binomial distribution may be a suitable model for the number of faulty bolts in the sample.
(2)

(b) Using a 5% significance level, find the critical region for a two-tailed test of the hypothesis that the probability of a bolt being faulty is $\frac{1}{4}$. The probability of rejection in either tail should be as close as possible to 0.025
(3)

(c) Find the actual significance level of this test.
(2)

In the sample of 50 the actual number of faulty bolts was 8.

(d) Comment on the company's claim in the light of this value. Justify your answer.
(2)

The machine making the bolts was reset and another sample of 50 bolts was taken. Only 5 were found to be faulty.

(e) Test at the 1% level of significance whether or not the probability of a faulty bolt has decreased. State your hypotheses clearly.
(6)
(Total 15 marks)

BINOMIAL DISTRIBUTION

Question 19

a) Binomial distribution because two outcomes (head or tail) and fixed probability ($p=0.5$). With $n=15$ trials and X the number of heads, we have: $X \sim B(15, 0.5)$

b) $P(X=8) = \frac{15!}{8!7!} 0.5^8 (1-0.5)^7 = \frac{15!}{8!7!} 0.5^{15} = 0.1964$.

OR $P(X=8) = P(X \leq 8) - P(X \leq 7)$
Using the table for $n=15$ and $p=0.5$, we find:
$P(X=8) = 0.6964 - 0.5 = 0.1964$.

c) $P(X \geq 4) = 1 - P(X \leq 3)$
$= 1 - 0.0176 = 0.9284$.

d) Let Y be the number of times the 2^{nd} coin shows head.
$Y \sim B(15, 0.5)$. $H_0: p=0.5$; $H_1: p > 0.5$
$P(X \leq 11) = 0.9824 \rightarrow P(X \geq 12) = 0.0176 = 0.01 + 0.0076$
$P(X \leq 12) = 0.9963$: closest value to 0.99 ($= 99\%$, ie $100\% - 1\%$) $\rightarrow P(X \geq 13) = 0.0037 = 0.01 - 0.0063$
CR is $X \geq 12$ (or $X \geq 13$)
13 is in the critical region, so there is evidence that the 2^{nd} coin is biased in favour of heads.

Question 6

a) H_0: $X \sim B(25, 0.2)$; $H_1: p \neq 0.2$

$P(X=0) = 0.0038$; $P(X \leq 1) = 0.0274 \rightarrow$ closest value to 0.025.
$\Rightarrow$ CR: $X \leq 1$.

$P(X \leq 9) = 0.9827 \rightarrow P(X > 9) = 0.0173 \rightarrow$ closest value to 0.025
$P(X \leq 8) = 0.9532 \rightarrow P(X > 8) = 0.0468$
$\Rightarrow$ CR $X > 9$

b) significance level = probability of incorrectly rejecting the null hypothesis:
$= 0.0274 + 0.0173 = 0.0447 = 4.47\%$

c) Let Y be the number of bowls with minor defects.
H_0: $Y \sim B(20, 0.20)$; $H_1: p < 0.20$ (we test whether the proportion of faulty bowls has decreased) $\Rightarrow$ one-tail test at a 10% level of significance.
$\Rightarrow P(Y \leq 1) = 0.0692 = 0.1 - 0.0308 \rightarrow$ closest value to 0.1 (10%)
$P(Y \leq 2) = 0.2061 = 0.1 + 0.1061$
$\rightarrow$ CR is $Y \leq 1$.
2 is not in the critical region, so there's no evidence that the proportion of defective bowls has decreased.

Question 14

a) two reasons among : • 2 outcomes (faulty / not faulty)
• constant probability
• independence
• fixed number of trials

b) $X \sim B(50, 0.25)$

H_0 ; $H_1: p \neq 0.25$ → two-tailed test with a 5% significance level

$P(X \leq 6) = 0.0194 = 0.025 - 0.0056$ → closest value to 0.025

$P(X \leq 7) = 0.0453 = 0.025 + 0.0203$

⇒ $CR_{low}: X \leq 6$

$P(X \leq 18) = 0.9713$ → $P(X > 18) = 0.0287$ → closest to 0.025

$P(X \leq 17) = 0.9861$ → $P(X > 17) = 0.0139$

⇒ $CR_{high}: X > 18$ or $X \geq 19$

⇒ CR : $X \leq 6$ and $X \geq 19$

c) significance level = $0.0194 + 0.0287 = 0.0481$

d) 8 is not in the critical region so no evidence to reject the claim of $p = 0.25$.

e) let Y be the number of faulty bolts after the machine reset.

$H_0: Y \sim B(50, 0.25)$; $H_1: p < 0.25$ → one tail test at 1%

$P(X \leq 5) = 0.0070 \leq 0.01$

OR, CR: X<6

→ there is evidence that the proportion of faulty bolts has decreased.